

# On the Chromatic Number of $\mathbb{R}^n$ for Small Values of $n$

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## Abstract

The lower bound for the chromatic number of  $\mathbb{R}^n$  is improved for  $n = 6, 7, 10, 11, 12, 13$  and  $14$ .

## 1 Introduction.

The chromatic number of  $n$ -dimensional Euclidean space, denoted  $\chi(\mathbb{R}^n)$ , is the minimum number of colors that can be assigned to the points of  $\mathbb{R}^n$  so that no two points at distance one receive the same color. In this note, we establish new lower bounds for  $\chi(\mathbb{R}^n)$  for several small values of  $n$ .

In [10], a table of lower bounds for the  $\chi(\mathbb{R}^n)$  was given. Besides the new bounds given in that paper, we are aware of only one other improvement [6]. Based on this improvements, we give a modified table below. The table also indicates the new bounds given here.

## 2 $\chi(\mathbb{R}^6)$

We construct a graph  $G_{175}$  of order 175 with chromatic number 12. The vertices of the graph are a set of points in  $\mathbb{R}^6$  generated by 11 special points. The coordinates of each of these special points is permuted in all  $6! = 720$  possible ways to obtain the full set of 175 vertices. The graph will be constructed as an

n	known bound	new bound
2	4 [12]	
3	6 [13]	
4	9 [6]	
5	9 [2]	
6	11 [3]	12
7	15 [16]	16
8	16 [19]	
9	21 [10]	
10	23 [10]	26
11	24 [10]	26
12	24 [11]	36
13	31 [19]	36
14	35 [19]	36
15	37 [19]	

Table 1: Lower bounds on  $\chi(\mathbb{R}^n)$  for small  $n$ .

$r$ -distance graph for  $r = \sqrt{8}$ . The coordinates for each point can be divided by  $r$  to obtain a unit distance graph.

The following table lists the aforementioned 11 special points along with the number of distinct points generated by permuting their coordinates. For each of these 11 points,  $v_i$ , let  $V_i$  denote the set of points obtained by permuting the coordinates, and let  $n_i = |V_i|$ , as shown in the table.

Point	Coordinates						$n_i$
$v_1$	0	0	0	0	0	0	1
$v_2$	2	2	0	0	0	0	15
$v_3$	2	2	2	2	0	0	15
$v_4$	$\sqrt{3}$	1	1	1	1	1	6
$v_5$	$\sqrt{3}$	1	1	1	1	-1	30
$v_6$	$-\sqrt{3}$	1	1	1	1	1	6
$v_7$	$-\sqrt{3}$	1	1	1	1	-1	30
$v_8$	$2 + \sqrt{3}$	1	1	1	1	1	6
$v_9$	$2 + \sqrt{3}$	1	1	1	1	-1	30
$v_{10}$	$2 - \sqrt{3}$	1	1	1	1	1	6
$v_{11}$	$2 - \sqrt{3}$	1	1	1	1	-1	30
							175

Table 2: The 11 points that generate the vertices of  $G_{175}$ .

Observe that the subgraphs induced by  $V_2$  and by  $V_3$  are each isomorphic to the line graph of  $K_6$ . In the case of  $V_2$  this is because two points in  $V_2$

are adjacent if their dot product is 4. This occurs when there is exactly one coordinate position where both points have a 2. In the case of  $V_3$ , two points are adjacent if there is exactly one coordinate position where both points have a zero. Next define an isomorphism  $\phi : V_2 \rightarrow V_3$  by letting

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6) = (2 - u_1, 2 - u_2, 2 - u_3, 2 - u_4, 2 - u_5, 2 - u_6).$$

Then the edges joining  $V_2$  and  $V_3$  are given as follows. A vertex  $x \in V_2$  is adjacent to a vertex  $y \in V_3$  whenever  $y$  is not adjacent to (or equal to)  $\phi(x)$ . This gives the subgraph  $H$  of the  $\sqrt{8}$ -distance graph induced by  $V_2 \cup V_3$ .

Note that the independence number  $\alpha(L(K_6)) = 3$ . However, in  $H$  an independent set of size three consisting of vertices from  $V_2$  dominates every vertex in  $V_3$ . So  $\alpha(H) = 4$ . In fact, it can be seen that every maximum independent set can be obtained from the following independent set by an appropriate permutation of coordinates

$$\begin{pmatrix} 2, & 2, & 0, & 0, & 0, & 0 \\ 0, & 0, & 2, & 2, & 0, & 0 \\ 2, & 0, & 2, & 0, & 2, & 2 \\ 2, & 0, & 0, & 2, & 2, & 2 \end{pmatrix}$$

Thus we have the following lemma.

**Lemma 2.1.** *The graph  $H$  has chromatic number 8.*

The proof of the following theorem can be completed by a short computer search that makes strong use of this lemma.

**Theorem 2.1.** *The graph  $G$  has chromatic number 12 and can be represented as a unit distance graph in  $\mathbb{R}^6$ .*

### 3 $\chi(\mathbb{R}^7)$

**Theorem 3.1.**  $\chi(\mathbb{Q}^7) \geq 16$ .

*Proof.* Consider the following fourteen sets in  $\mathbb{Q}^7$ :

$$\begin{aligned} S_{123} &= [\pm 2, \pm 2, \pm 2, 0, 0, 0, 0], & T_{123} &= [0, 0, 0, \pm 1, \pm 1, \pm 1, \pm 1], \\ S_{145} &= [\pm 2, 0, 0, \pm 2, \pm 2, 0, 0], & T_{145} &= [0, \pm 1, \pm 1, 0, 0, \pm 1, \pm 1], \\ S_{167} &= [\pm 2, 0, 0, 0, 0, \pm 2, \pm 2], & T_{167} &= [0, \pm 1, \pm 1, \pm 1, \pm 1, 0, 0], \\ S_{247} &= [0, \pm 2, 0, \pm 2, 0, 0, \pm 2], & T_{247} &= [\pm 1, 0, \pm 1, 0, \pm 1, \pm 1, 0], \\ S_{256} &= [0, \pm 2, 0, 0, \pm 2, \pm 2, 0], & T_{256} &= [\pm 1, 0, \pm 1, \pm 1, 0, 0, \pm 1], \\ S_{346} &= [0, 0, \pm 2, \pm 2, 0, \pm 2, 0], & T_{346} &= [\pm 1, \pm 1, 0, 0, \pm 1, 0, \pm 1], \\ S_{357} &= [0, 0, \pm 2, 0, \pm 2, 0, \pm 2], & T_{357} &= [\pm 1, \pm 1, 0, \pm 1, 0, \pm 1, 0]. \end{aligned}$$

Denote

$$\begin{aligned} S &= S_{123} \cup S_{145} \cup S_{167} \cup S_{247} \cup S_{256} \cup S_{346} \cup S_{357}, \\ T &= T_{123} \cup T_{145} \cup T_{167} \cup T_{247} \cup T_{256} \cup T_{346} \cup T_{357}. \end{aligned}$$

Let  $G$  be the graph whose vertices are the points in  $S \cup T$ . Two vertices are adjacent if and only if their distance is 4. It can be checked that  $G$  has 168 vertices and 4396 edges. We are going to prove that  $\chi(G) = 16$ .

Let  $H$  be the subgraph of  $G$  induced by the points in  $S$ . One can verify that  $H$  is a graph of order  $|V(H)| = 56$ , size  $|E(H)| = 756$ , independence number  $\alpha(H) = 4$  and chromatic number  $\chi(H) = |V(H)|/\alpha(H) = 14$ .

Similarly, let  $K$  be the subgraph of  $G$  induced by the points in  $T$ . One can verify that  $K$  is a matching of order  $|V(K)| = 112$ , size  $|E(K)| = 56$ , independence number  $\alpha(K) = 56$  and chromatic number  $\chi(K) = |V(K)|/\alpha(K) = 2$ . It follows that  $\chi(G) \leq \chi(H) + \chi(K) = 14 + 2 = 16$ .

Let  $M$  be an independent set of  $G$ . From the observation above  $|M \cap V(H)| \leq 4$ . We say that  $M$  is an *independent set of type  $k$*  if  $|M \cap V(H)| = k$  for some  $0 \leq k \leq 4$ . The following claim can be easily checked

**Claim 3.1.** *Let  $M$  be an independent set of type  $k$  in  $G$ . Then, the following hold:*

$$\begin{aligned} \text{If } k = 0, \quad \text{then } |M \cap V(K)| &\leq 56. \\ \text{If } k = 1, \quad \text{then } |M \cap V(K)| &\leq 24. \\ \text{If } k = 2, \quad \text{then } |M \cap V(K)| &\leq 24. \\ \text{If } k = 3, \quad \text{then } |M \cap V(K)| &\leq 3. \\ \text{If } k = 4, \quad \text{then } |M \cap V(K)| &\leq 3. \end{aligned}$$

Suppose that  $\chi(G) \leq 15$ . Then the set of vertices of  $G$  can be partitioned into 15 independent sets. Denote by  $m_k$  the number of independent sets of type  $k$  in this partition,  $0 \leq k \leq 4$ . Then from Claim 3.1, the following relations hold true:

$$\begin{aligned} 15 &= m_0 + m_1 + m_2 + m_3 + m_4. \\ 56 &= m_1 + 2m_2 + 3m_3 + 4m_4. \\ 112 &\leq 56m_0 + 24m_1 + 24m_2 + 3m_3 + 3m_4. \end{aligned}$$

But it is easy to check (either by hand or a short program) that this system has no solutions in nonnegative integers. Thus  $\chi(G) \geq 16$ .  $\square$

## 4 $\chi(\mathbb{R}^{10})$

The construction for  $\chi(\mathbb{R}^{10})$  is related to the well-known Frankl-Wilson construction [7] which established an exponential lower bound for  $\chi(\mathbb{R}^{10})$ , and also gives the best constructive lower bound for classical diagonal Ramsey numbers.

The vertices of this graph are identified with the points  $(x_1, x_2, \dots, x_{11})$  in  $\mathbb{R}^{11}$  such that each  $x_i = 0$  or 1, and

$$\sum_{i=1}^{11} x_i = 5$$

There are  $\binom{11}{5} = 462$  such points. Two points are adjacent if their distance is 2 (and so their Hamming distance is 4). This graph is regular of degree  $\binom{5}{3}\binom{6}{2} = 150$ .

A computation reveals that the independence number of  $G$ , denoted as usual by  $\alpha(G)$ , is 18. We did this computation two ways. First, our own special program written for graph of this type was used. Second, the result was verified by the `mcqd` program of Konc and Janežič [9]. Using the fact that  $\chi(G) \geq \frac{n}{\alpha G}$ , for any graph  $G$  of order  $n$ , we find that  $\chi(G) \geq \lceil \frac{462}{18} \rceil = 26$ .

Finally, for each point, the sum of the coordinates is 5, so the points are located on a 10-dimensional hyperplane, so we have the following.

**Theorem 4.1.**  $\chi(\mathbb{R}^n) \geq 26$ , for  $n = 10, 11$ .

## 5 $\chi(\mathbb{R}^{12})$

The construction for  $\mathbb{R}^{12}$  parallels that of  $\mathbb{R}^{10}$ . In this case the vertex set consists of all 0–1-vectors in  $\mathbb{R}^{13}$  with Hamming weight 6. Again two vertices are adjacent if their distance is 2 (Hamming distance 4). So the the graph has order  $\binom{13}{6} = 1716$  and degree  $\binom{13}{6}\binom{6}{2}$ .

For this case, computing the independence number is a much harder computation. But again, both `mcqd` and our specialized program were able to determine that the independence number is 46. Hence the chromatic number is at least  $\lceil \frac{1716}{46} \rceil = 36$ .

**Theorem 5.1.**  $\chi(\mathbb{R}^n) \geq 36$ , for  $n = 12, 13, 14$ .

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